Spectral Patching for Non-Stationary Linear Inversion Problems

Adelson S. de Oliveira, PETROBRAS S/A, Brazil

This paper was prepared for presentation at the 14th International Congress of the Brazilian Geophysical Society, held in Rio de Janeiro, Brazil, August 3-6, 2015.

Abstract

This paper deals with using causality as *a priori* information to overcome the difficulties associated with the compensation of source and receiver ghosts in seismic data. A mathematical approach to a rather general linear inverse problem is presented followed by some specific synthetic examples that makes it clear its shortcomings and points to new strategies to deal with the inherent uncertainties on the process model parameters.

Introduction

Many seismic processes are mathematically described by ill-posed or ill-conditioned linear equations. Often, the solution to such equations is known only up to the extent of the associated null space of the operator that characterizes the problem. At this point, different *a priori* information are gathered so as to reduce the uncertainties and increase stability in the solution estimate.

Causality is intrinsic to almost all seismic processes and represent a large source of information that may sometimes "heal" (as a spectral patch) some of the common difficulties in inverting seismic problems. Causality implies as many additional equations or constraints to a linear inversion problem as negative times are used. Thus, at least in principle, a large amount of ill-conditioned problems may have indeterminacy reduced. Requiring causality is analogous to require compact support and what is discussed below could be easily extended to accomplish this broader condition.

A formal expression for the solution of a general causal linear problem like,

$$A\,x = b \qquad (1)$$

that satisfies $x = 0$ for time $t < 0$, may be written so that causality is strictly considered. Rewriting A as

$$A = V \Lambda U^H \qquad (2)$$

where U and V are orthogonal matrices and Λ is diagonal, the inverse of A is defined in terms of the inverse of Λ. Defining $\tilde{b} = V^H b$, attaching subscripts s and u standing for, respectively, "stable" and "unstable", to the inverse of the elements λ of Λ according to $\lambda_s \geq \mathrm{threshold} > \lambda_u$, one may write,

$$\begin{vmatrix} x \\ 0 \end{vmatrix} = \begin{vmatrix} U_s^> & U_u^> \\ U_s^< & U_u^< \end{vmatrix} \begin{vmatrix} \Lambda_s^{-1} & 0 \\ 0 & \Lambda_u^{-1} \end{vmatrix} \begin{vmatrix} \tilde{b}_s \\ \tilde{b}_u \end{vmatrix} \qquad (3)$$

where superscripts $>$ and $<$, respectively, stands for $t > 0$ and $t < 0$.

From the $t < 0$ part of equation (3) above, it is possible to write,

$$\overline{\Lambda_u^{-1}\tilde{b}_u} = -\left[U^{<H}_u U^<_u\right]^{-1} U^{<H}_u U^<_s \Lambda_s^{-1} \tilde{b}_s \qquad (4)$$

which allows to estimate, in a least square sense, the pseudo-inverse of the unstable (unknown) components of A in terms of its well-behaved subspace. This in turn yields the following expression for the solution x,

$$x = \left\{U_s^> \Lambda_s^{-1} - U_u^> \left[U^{<H}_u U^<_u\right]^{-1} U^{<H}_u U^<_s \Lambda_s^{-1}\right\} \tilde{b}_s \qquad (5)$$

which depends only on the stable part of the operator A.

The Stationary Case

The case where the operator A is stationary (A is a Toeplitz and circular matrix) is of particular interest (Oliveira, 2014). In this case, A can be diagonalized with Fourier transforms and expression (2) can be read as,

$$A = F^{-1} \Lambda F \qquad (6)$$

where F is the discrete Fourier transform. This case is much simpler and less expensive computationally since only fast Fourier transforms are required.
Thus, for a stationary matrix A, expression (5) can be seen as a sequence of operations given by:

1. Fourier transform the right hand side of equation (1) to get $\tilde{b}$;
2. Fourier transform the inner convolution operator in A and define the stable and unstable spectral parts;
3. Accordingly slice Fourier inversion matrix so as stable, unstable, positive and negative times, are identified and estimate the patch as given in (4);
4. Finally combine the estimated patch with stable part of the solution as in (5) to get x.

Ghosts

Source and receiver's ghosts are present when sources and/or receivers are placed close enough to a highly reflective interface to produce a secondary arrival at a short time interval as compared to the seismic pulse extension. In this case, all reflections seems "blurred" with smaller resolution. In the Fourier domain, the ghost operator is given by,

$$G(\omega, t_g) = 1 - r_0 \mathrm{e}^{-i\omega t_g} \qquad (7)$$

where t_g and ω are, respectively, the ghost time interval and the angular frequency, and r_0 is the ghost generating interface reflection coefficient.

When the offset between source and receivers are not small as compared to the depth of the reflectors of interest, the ghost time interval varies with the observation time t. The ghost time interval is smaller for shallower events and tends to $t_g = 2z_g/v$ as depth increases, z_g is the source/receiver distance to ghost generating interface. The ghost operator tends asymptotically to be stationary for deeper events but shallower events (greater offsets) may deserve special attention.

For the sake of simplicity, let's write the relation between seismic traces with and without ghost in the following way,

$$S_g(\omega) \;=\; G(\omega, t_g(t))\, s(t) \tag{8}$$

with $s(t)$ the trace without ghost in time-domain, $S_g(\omega)$ the trace with ghost in the frequency domain, and $G(\omega, t_g)$ a matrix with entries given by:

$$G_{m,n} \;=\; \left[1 - r_0 \mathrm{e}^{-i\omega_m t_g(t_n)}\right] \mathrm{e}^{-i\omega_m t_n}$$

for a given function $t_g(t)$.

Expression (8) above can be cast in the form of (2) if one writes $G = W\Lambda U^H$ and apply a Fourier transform:

$$s_g \;=\; FGs \;=\; FW\Lambda U^H s \;=\; V\Lambda U^H s \tag{9}$$

where FW was rewritten as another (also orthogonal) matrix V.

Synthetic examples

A synthetic trace with source-receiver offset of 3000 m, constant velocity of 2000 m/s and reflections set so as to appear at constant 0.2 s time interval is considered. The source and receiver distances to the ghost generating interface are, respectively, 0 and 30 meters and the ghosts are generated with a reflection coefficient of $r_0 = -1$. The ghost time intervals at the registered events vary from 10 to 27 milliseconds for corresponding notch frequencies of about 95 to 36 Hz. Figure 1 shows the seismic pulse and its frequency spectrum. Figure 2 shows a seismic trace with ghosts at different events and its corresponding frequency content. The diversity of notches yields an apparently notch free trace. Regular seismic process would attempt to compensate for a single (stationary) ghost usually yielding severe stability problems.

The question of how to determine the correct ghost operator (it depends on sea surface waves, tidal effects, etc.) is a major concern. However, this is not treated in this paper. This paper is rather concerned with basics in the use of causality to help defining a physically acceptable solution.

Figure 3 brings a comparison between the solution components $x_s = U_s^{>}\Lambda_s^{-1}\tilde{b}_s$ and $x_u = U_u^{>}\overline{\Lambda_u^{-1}\tilde{b}_u}$, respectively, the stable component and the unstable complement as given in (4). Figure 3 indicates that the lack of information due to a cut in the spectrum of the ghost operator can be perfectly covered by the causality property. Figure 4 shows the comparison between the original trace and the result of $x_s + x_u$ which represents the compensation for ghosts using causality. Figure 5 has the spectrum of the result to be compared to the spectra shown at figure 1 (the pulse spectrum) and figure 2 (the original trace with ghosts).

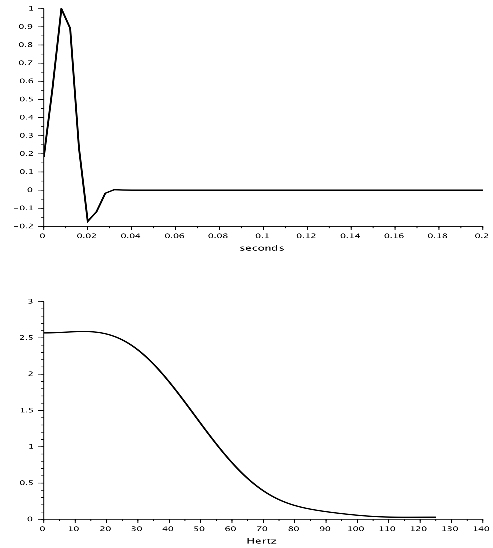

Figure 1: *From top to bottom: seismic pulse, seismic pulse frequency spectrum.*

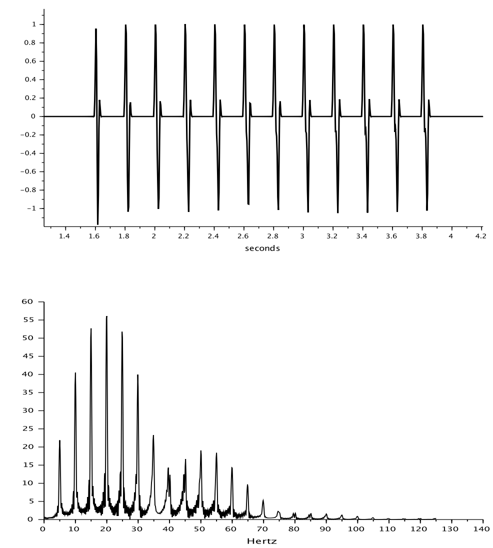

Figure 2: *From top to bottom: seismic trace, seismic trace frequency spectrum*

Noise

The basic assumption of a causal trace after applying the (correct) inverse of a linear operator is expected to hold for the signal. Noise may not fill this requirement. The inverse operator is not expected to produce a causal result even if causal noise is considered since, by definition, noise is not supposed to be part of the underlying physical model describing the system.

Causality is not clearly apparent in a real seismic trace. In fact, typically, sampling begins at $t = 0$ and no information about the noise before recording is available. To use causality, one must pad real traces with zeros for negative

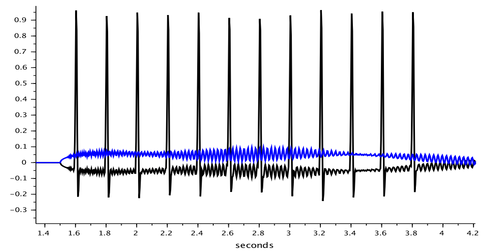

Figure 3: *Black: the stable component; Blue: the unstable (or complement to causal) component.*

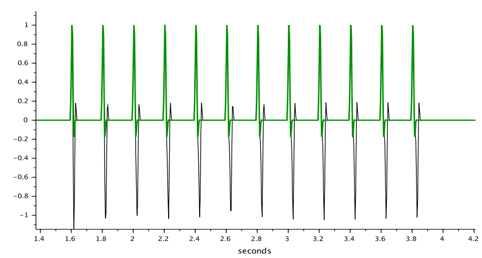

Figure 4: *Black: the original trace with ghosts; Green: the resulting trace after summing stable and unstable components.*

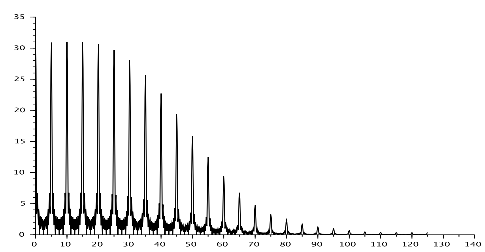

Figure 5: *The Fourier spectrum of the result. It can be described as the spectrum of a "comb" of spikes modulated by the pulse spectrum, as expected.*

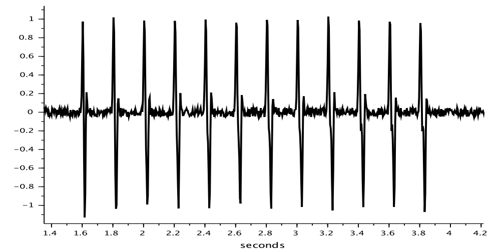

Figure 6: *The input trace with random noise. Note the apparent signal to noise ratio for comparison with the resulting trace after inversion and spectral patching shown at figure 7.*

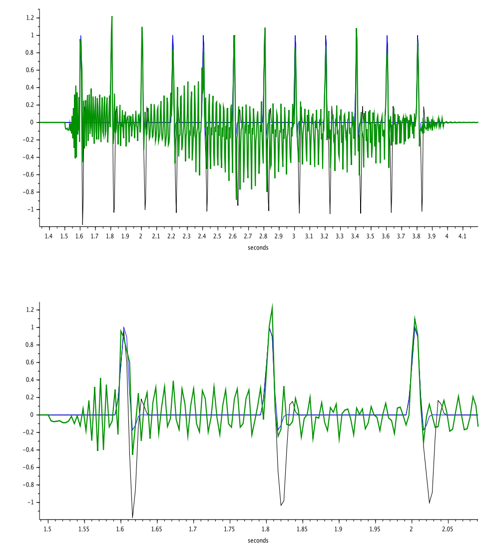

Figure 7: *The result of inverting with the causal complement as a spectral patching to the trace given at figure 6. Top: a broader window; Bottom: a zoom over shorter times. In both figures, Black: the original trace with ghost and no noise; Blue: the desired result with no ghost (without noise); and Green: the result without ghost but with amplified noise.*

times. This makes every event causal until the inverse filter is applied. After inversion, only events that fill in the model is expected to remain causal.

In Oliveira (2015), it is discussed a trade-off between the selection of the unstable portion, the amount of random noise, and the quality of the solution for the stationary case. It is interesting to notice that defining a shorter unstable region produces a greater amount of noise since one gets closer to the singularities of the ghost operator. On the other hand, a rather big unstable region prevents the method from using the reliable information to compute the spectral patching or causal complement to the solution. At some extent, these considerations are valid for the non-stationary case. However, since notches may not be evident due to its diversity, the consequences of a smaller unstable region might not be that clear.

Another important aspect to consider when noise is present is that the spectral patching as defined in (4) is the least square norm causal complement. It is not unique. It is defined after the pseudo-inverse of $[U^{<H}_u U^{<}_u]$ is computed and this depends on a definition of another "stable region". In figure 3, this pseudo-inverse was computed with a rather small tolerance for the pseudo-inverse, for conceptual reasons. This can not be made if a relatively big amount of noise is present.

Figure 6 shows the input trace with random noise. Figure 7 has the result of the process applied to the trace at figure 6. The amount of noise has increased remarkably. For a moderate amount of noise this effect is not expected to be so clear but it will still be there and has to be avoided.

Summary and Conclusions

An alternative method to handle difficulties in ill-posed/ill-conditioned non-stationary linear inversion problems based on the causality of the solution was presented. A Practical example of source and receiver ghosts in seismic data was discussed. Limitations and perspectives for reaching a broader frequency spectrum were addressed with synthetic data with and without random noise. According to what was shown, a careful control of the extent of the unstable zone to be patched is fundamental to accomplish a less noisy and more precise result. Another fundamental aspect in this procedure, not addressed here, is the definition of the ghost operator itself which has a direct impact on the basic hypothesis of a causal solution.

References

Golub, Gene H.; Van Loan, Charles F. (1996). Matrix Computations (3rd ed.). Johns Hopkins. ISBN 978-0-8018-5414-9.

Oliveira, A. S. (2014); Spectral Patching, Second EAGE/SBGf Workshop, Broadband Seismic, Rio de Janeiro, Brazil.

Acknowledgements

I would like to thank PETROBRAS for permission to publish this work.